Math 522 Exam 9 Solutions

1. Calculate $\big((\mu \star d) \star (d \star \phi)\big)(10)$.

   It would be very messy to solve this problem without using some of our simplifying theorems. Because $\star$ is associative and commutative, we can drop the parentheses and rearrange at will. $\mu \star d \star d \star \phi = \mu \star 1 \star 1 \star d \star \phi = \epsilon \star 1 \star d \star \phi = 1 \star \phi \star d = Id \star d$. Now, $(Id \star d)(10) = Id(1)d(10) + Id(2)d(5) + Id(5)d(2) + Id(10)d(1) = 1 \times 4 + 2 \times 2 + 5 \times 2 + 10 \times 1 = 28$.

   Or, if you prefer, $Id \star d = Id \star 1 \star 1 = \sigma \star 1$, and $(\sigma \star 1)(10) = \sigma(1)1(10) + \sigma(2)1(5) + \sigma(5)1(2) + \sigma(10)1(1) = 1 \times 1 + 3 \times 1 + 6 \times 1 + 18 \times 1 = 28$.

2. Solve the congruence $7x^{9,876,543,210,123,456,789} \equiv 9 \pmod{50}$.
   HINT: Mod 50, $3^1 \equiv 3, 3^2 \equiv 9, 3^3 \equiv 27, 3^4 \equiv 31, 3^5 \equiv 43, 3^6 \equiv 29, 3^7 \equiv 37, 3^8 \equiv 11, 3^9 \equiv 33, 3^{10} \equiv 49, 3^{11} \equiv 47, 3^{12} \equiv 41, 3^{13} \equiv 23, 3^{14} \equiv 19, 3^{15} \equiv 7, 3^{16} \equiv 21, 3^{17} \equiv 13, 3^{18} \equiv 39, 3^{19} \equiv 17, 3^{20} \equiv 1$.

   BONUS: Find all primitive roots modulo 50.

   $\phi(50) = \phi(2)\phi(25) = 20$, and the hint shows that 3 belongs to 20, so 3 is a primitive root. It is not necessary (but perhaps convenient) to arrange the hint into a table of indices.

Number	1	3	7	9	11	13	17	19	21	23	27	29	31	33	37	39	41	43	47	49
Index	20	1	15	2	8	17	19	14	16	13	3	6	4	9	7	18	12	5	11	10

   $ind\, 9 \equiv ind\,(7x^{9,876,543,210,123,456,789}) \equiv ind\, 7 + 9,876,543,210,123,456,789\; ind\, x \equiv ind\, 7 + 9\; ind\, x \pmod{20}$. Hence $2 \equiv 15 + 9\; ind\, x$, or $7 \equiv 9\; ind\, x \pmod{20}$. We now need to find $9^{-1} \pmod{20}$, which is actually 9 again. We multiply by 9 on both sides, finding $3 \equiv 63 \equiv 7 \times 9 \equiv (9 \times 9)\; ind\, x \equiv ind\, x \pmod{20}$. Hence $x = 27$ is the unique solution, modulo 50.

   BONUS: The integers in $[1, 20]$ relatively prime to 20 are $1, 3, 7, 9, 11, 13, 17, 19$ (there are $\phi(20) = 8$ of them). We know 3 is a primitive root, so by Cor. 7.1, these powers of 3 are also primitive roots. Namely, $3, 27, 37, 33, 47, 23, 13, 17$ are the primitive roots of 50.

3. High score=92, Median score=65, Low score=50